

# Two weighted estimates for generalized fractional maximal operators on non homogeneous spaces.

Gladis Pradolini and Jorgelina Recchi

## Abstract

Let  $\mu$  be a non-negative Borel measure on  $\mathbb{R}^d$  satisfying that the measure of a cube  $\mathbb{R}^d$  is smaller than the length of its side raised to the  $n$ -th power,  $0 < n \leq d$ .

In this article we study the class of weights related to the boundedness of radial fractional type maximal operator associated to a Young function  $B$  in the context of non-homogeneous spaces related with the measure  $\mu$ . This type of maximal operators are the adequate operators related with commutators of singular and fractional operators. Particularly, we give an improvement of a two weighted result for certain fractional maximal operator proved in [26].

## 1 Introduction and statements of the main results

Let  $\mu$  be a non-negative upper Ahlfors  $n$ -dimensional measure on  $\mathbb{R}^d$ , that is, a Borel measure satisfying

$$\mu(Q) \leq l(Q)^n \quad (1.1)$$

for any cube  $Q \subset \mathbb{R}^n$  with sides parallel to the coordinate axes, where  $l(Q)$  stands for the side length of  $Q$  and  $n$  is a fixed real number such that  $0 < n \leq d$ . Besides, for  $r > 0$ ,  $rQ$  will mean the cube with the same centre as  $Q$  and with  $l(rQ) = rl(Q)$ .

In the last decades, this measure have proved to be adequate for the development of many results in Harmonic Analysis which were known that hold in the context of doubling measures, that is, Borel measures  $\nu$  for which there exists a positive constant  $D$  such that  $\nu(2Q) \leq D\nu(Q)$  for every cube  $Q \subset \mathbb{R}^d$ . For example, many interesting results related with different operators and spaces of functions with non doubling measures can be found in [15], [16], [13], [24], [7] and [14] between a vast bibliography on this topic.

In [26] the authors studied two weighted norm inequalities for a fractional maximal operator associated to a measure  $\mu$  satisfying condition (1.1). Concretely, they considered

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the following version of the fractional maximal operator defined, for  $0 \leq \alpha < 1$ , by

$$M_\alpha f(x) = \sup_{Q \ni x} \frac{1}{\mu(5Q)^{1-\alpha}} \int_Q |f(y)| d\mu(y),$$

and proved the following result.

**Theorem 1.1** *Let  $1 < p < q < \infty$  and  $0 \leq \alpha < 1$ . Let  $(u, v)$  be a pair of weights such that for every cube  $Q$*

$$l(Q)^{n(1-1/p)} \mu(Q)^{\alpha-1} u(3Q)^{\frac{1}{q}} \|v^{-\frac{1}{p}}\|_{\Phi, Q} \leq C \quad (1.2)$$

where  $\Phi$  is a Young function whose complementary function  $\bar{\Phi} \in B_p$ . Then

$$\left( \int_{\mathbb{R}^d} M_\alpha f(x)^q u(x) d\mu(x) \right)^{1/q} \leq C \left( \int_{\mathbb{R}^d} |f(x)|^p v(x) d\mu(x) \right)^{1/p}.$$

for every  $f \in L^p(v)$  bounded with compact support.

The radial Luxemburg type average in theorem above is defined by

$$\|\cdot\|_{\Phi, Q} = \inf \{ \lambda > 0 : \frac{1}{l(Q)^n} \int_Q \Phi \left( \frac{|f(x)|}{\lambda} \right) d\mu(x) \leq 1 \},$$

and a Young function  $B$  satisfies the  $B_p$  condition,  $1 < p < \infty$ , if there is a positive constant  $c$  such that

$$\int_c^\infty \frac{B(t)}{t^p} \frac{dt}{t} < \infty.$$

Let us make some comments about Theorem 1.1. When  $\mu$  is the Lebesgue measure and  $u = v = 1$ , it is easy to note that condition (1.2) holds if and only if  $1/q = 1/p - \alpha$  for any  $\Phi$  as in the hypothesis. On the other hand, if we consider an upper Ahlfors  $n$ -dimensional measure  $\mu$  and if we take  $\Phi(t) = t^{rp'}$ , for  $1 < r < \infty$ ,  $1/q = 1/p - \alpha$  and  $u = v = 1$  in condition (1.2) we have that if the following inequality holds

$$l(Q)^{n(1-1/p)} \mu(Q)^{\alpha-1} \mu(3Q)^{\frac{1}{p}-\alpha} \left( \frac{\mu(Q)}{l(Q)^n} \right)^{1/rp'} \leq C,$$

then

$$\left( \frac{l(Q)^n}{\mu(Q)} \right)^{1/(p'r')} \leq C$$

which implies that the measure  $\mu$  satisfying the growth condition (1.1) also satisfies the “lower” case, that is  $\mu(Q) \geq Cl(Q)^n$ , with a constant independent of  $Q$ . So, the weights  $u = v = 1$  are not allowed in this case unless the measure is Ahlfors, that is  $\mu(Q) \simeq l(Q)^n$ , for every cube  $Q$ . Moreover, let

$$Mu(x) = \sup_{Q \ni x} \frac{1}{\mu(Q)} \int_Q |u(y)| d\mu(y).$$

When  $\mu$  is the Lebesgue measure and  $\Phi(t) = t^{rp'}$ , it is easy to check that the pair of weights  $(u, (Mu)^{p/q})$ , with  $1/q = 1/p - \alpha$ , satisfies condition (1.2). On the other hand, suppose that this pair satisfies the same condition for a measure satisfying (1.1) and let  $u \in A_1(\mu)$ . Thus, the following chains of inequalities holds

$$\begin{aligned}
C &\geq \frac{l(Q)^{n/p'}}{\mu(Q)^{1-\alpha-1/q}} \left( \frac{1}{\mu(Q)} \int_Q u \, d\mu \right)^{1/q} \left( \frac{1}{l(Q)^n} \int_Q ((Mu)^{p/q})^{-rp'/p} \right)^{1/(rp')} \\
&\geq \frac{l(Q)^{n/p'-n/(rp')}}{\mu(Q)^{1/p'-1/(rp')}} \left( \frac{1}{\mu(Q)} \int_Q u^{p/q} \, d\mu \right)^{1/p} \left( \frac{1}{\mu(Q)} \int_Q (u^{p/q})^{-rp'/p} \right)^{1/(rp')} \\
&\geq \frac{l(Q)^{n/(r'p')}}{\mu(Q)^{1/(r'p')}} \left( \frac{1}{\mu(Q)} \int_Q u^{p/q} \, d\mu \right)^{1/p} \left( \frac{1}{\mu(Q)} \int_Q (u^{p/q})^{-p'/p} \right)^{1/(p')} \\
&\geq \frac{l(Q)^{n/(r'p')}}{\mu(Q)^{1/(r'p')}} ,
\end{aligned}$$

which implies again that  $\mu$  must be an Ahlfors measure.

In [7] the authors considered the radial fractional maximal function associated to an upper Ahlfors  $n$ -dimensional measure  $\mu$  which is defined, for  $0 \leq \alpha < n$ , by

$$\mathcal{M}_\alpha f(x) = \sup_{Q \ni x} \frac{1}{l(Q)^{n-\alpha}} \int_Q |f(y)| \, d\mu(y).$$

In the same article they study weighted boundedness properties for  $\mathcal{M}_\alpha$  on non homogeneous spaces.

In this paper we introduce a generalized version of the radial fractional maximal operator defined in [7], associated to a Young function  $B$ . This type of maximal operators are not only a generalization but also they have proved to be the adequate operators related with commutators of singular and fractional integral operators in different settings, (see for example [2], [18], [19], [20], [21], [11], [12] and [3]). It is important to note that the examples of weights given above satisfy the condition obtained in our theorem when  $\mu$  is an upper Ahlfors  $n$ -dimensional measure. In this sense, when  $B(t) = t$ , our result is better than the corresponding result in [26].

In order to state the main results we introduce some preliminaries. Given a Young function  $B$ , we define  $L_\mu^B(\mathbb{R}^d)$  as the set of all measurable functions  $f$  for which there exists a positive number  $\lambda$  such that

$$\int_{\mathbb{R}^d} B\left(\frac{|f(x)|}{\lambda}\right) \, d\mu(x) < \infty.$$

The radial fractional type maximal operator associated to a Young function  $B$  is defined by

$$\mathcal{M}_{\alpha,B}(f)(x) = \sup_{Q \ni x} l(Q)^\alpha \|f\|_{B,Q}, \quad 0 \leq \alpha < n,$$

where

$$\|f\|_{B,Q} = \inf\{\lambda > 0 : \frac{1}{l(Q)^n} \int_Q B\left(\frac{|f(x)|}{\lambda}\right) \, d\mu(x) \leq 1\}$$

is the radial Luxemburg average (see §2.1). When  $B(t) = t$  then

$$\|f\|_{B,Q} = \frac{1}{l(Q)^n} \int_Q |f| d\mu. \quad (1.3)$$

When  $\alpha = 0$ , we write  $\mathcal{M}_{0,B} = \mathcal{M}_B$ .

The following theorem gives sufficient conditions for strong type inequalities for  $\mathcal{M}_{\alpha,B}$  on non homogeneous spaces.

**Theorem 1.2** *Let  $1 < p < q < \infty$ ,  $0 \leq \alpha < n$  and let  $\mu$  be an upper Ahlfors  $n$ -dimensional measure in  $\mathbb{R}^d$ . Let  $B$  be a submultiplicative Young function such that  $B^{q_0/p_0} \in B_{q_0}$  for some  $1 < p_0 \leq n/\alpha$  and  $1/q_0 = 1/p_0 - \alpha/n$ , and let  $\phi$  and  $\varphi$  be two Young functions such that  $C_1\varphi^{-1}(t)t^{\alpha/n} \leq B^{-1}(t) \leq C_2\phi^{-1}(t)t^{\alpha/n}$  for some positive constants  $C_1$  and  $C_2$ . If  $A$  and  $C$  are two Young functions such that  $A^{-1}C^{-1} \preceq B^{-1}$  with  $C \in B_p$  and  $(u, v)$  is a pair of weights such that for every cube  $Q$*

$$l(Q)^{\alpha - \frac{n}{p}} u(3Q)^{\frac{1}{q}} \|v^{-\frac{1}{p}}\|_{A,Q} \leq K \quad (1.4)$$

then, for all  $f \in L_\mu^p(v)$ .

$$\|\mathcal{M}_{\alpha,B}(f)\|_{L_\mu^q(u)} \leq C \|f\|_{L_\mu^p(v)},$$

**Remark 1.3** *When  $u = v = 1$  and  $1/q = 1/p - \alpha/n$  then condition (1.4) is satisfied for any upper Ahlfors  $n$ -dimensional measure  $\mu$ . Thus, this result is an improvement of that given in [26] in the sense that the unweighted boundedness of the operator is obtained for any measure satisfying the growth condition (1.1). The same is true for the second example considered above.*

**Remark 1.4** *When  $B(t) = t \log(e + t)^k$  it can be easily seen that  $B$  is submultiplicative,  $B^{q_0/p_0} \in B_{q_0}$  for every  $p_0, q_0 > 1$  and*

$$B^{-1}(t) \approx t^{\alpha/n} \frac{t^{1-\alpha/n}}{\log(e + t)^k} \approx t^{\alpha/n} \phi^{-1}(t),$$

when  $\phi(t) = (t \log(e + t)^k)^{\frac{n}{n-\alpha}}$ . Moreover, the functions  $A(t) = t^{rp'}$  and  $C(t) = (t \log(e + t)^k)^{(rp')'}$  satisfy

$$A^{-1}C^{-1} \preceq B^{-1}.$$

For  $\delta > 0$ , other examples are given by  $A(t) = t^{p'} \log(e + t)^{(k+1)p' - 1 + \delta}$  and  $C(t) = t^p \log(e + t)^{-(1+\delta(p-1))}$ , (see [6]).

It is also important to note that Theorem 3.1 in [7] is a special case of the previous theorem by considering  $A(t) = t^{rp'}$ ,  $C(t) = t^{(rp')'}$  and  $B(t) = t$ .

Let us make some comments about the upper Ahlfors  $n$ -dimensional measure  $\mu$  satisfying (1.1). It is well known that for such measures the Lebesgue differentiation theorem holds; that is, for every  $f \in L^1_{loc}(\mathbb{R}^d)$  and a.e.  $x$

$$\frac{1}{\mu(Q)} \int_Q f(y) d\mu(y) \rightarrow f(x),$$

when  $Q$  decreases to  $x$  (see [25]). However, if we take radial averages like those defined in (1.3) this is not longer true. In fact, let us consider  $\mu$  defined by  $d\mu(t) = e^{-t^2} dt$ , which is an upper Ahlfors 1-dimensional measure, and  $f(t) = e^{\theta t^2}$ ,  $\theta \in \mathbb{R}$ . Let  $x \in \mathbb{R}$ , then

$$\lim_{r \rightarrow 0} \frac{1}{2r} \int_{x-r}^{x+r} f(t) d\mu(t) = \lim_{r \rightarrow 0} \frac{1}{2r} \int_{x-r}^{x+r} e^{(\theta-1)t^2} dt = e^{(\theta-1)x^2}$$

which differs from  $f$  in a.e.  $x$ .

Given a Young function  $B$ , let  $h_B$  be the function defined by

$$h_B(s) = \sup_{t>0} \frac{B(st)}{B(t)}, \quad 0 \leq s < \infty.$$

If  $B$  is submultiplicative then  $h_B \approx B$ . More generally, given any  $B$ , for every  $s, t \geq 0$ ,  $B(st) \leq h_B(s)B(t)$ . It is easy to proof (see [4, Lemma 3.11]), that if  $B$  is a Young function then  $h_B$  is nonnegative, submultiplicative, increasing in  $[0, \infty)$ , strictly increasing in  $[0, 1]$  and  $h_B(1) = 1$ .

The following theorem gives an modular endpoint estimate for  $\mathcal{M}_{\alpha,B}$  on non-homogeneous spaces. When  $\mu$  is the Lebesgue measure, this result was proved in [5].

**Theorem 1.5** *Let  $0 \leq \alpha < n$  and let  $\mu$  an upper Ahlfors  $n$ -dimensional measure on  $\mathbb{R}^d$ . Let  $B$  be a Young function and suppose that, if  $\alpha > 0$ ,  $B(t)/t^{\frac{n}{\alpha}}$  is decreasing for all  $t > 0$ . Then there exists a constant  $C$  depending only on  $B$  such that for all  $t > 0$ ,  $\mathcal{M}_{\alpha,B}$  satisfies the following modular weak-type inequality*

$$\phi \left[ \mu \left( \{x \in \mathbb{R}^d : \mathcal{M}_{\alpha,B}(f)(x) > t\} \right) \right] \leq C \int_{\mathbb{R}^n} B \left( \frac{|f(y)|}{t} \right) d\mu(y),$$

for all  $f \in L^B_\mu(\mathbb{R}^d)$ , where  $\phi$  is any function such that

$$\phi(s) \leq C_1 \phi_1(s) = \begin{cases} 0 & \text{if } s = 0 \\ \frac{s}{h_B(s^{\alpha/n})} & \text{if } s > 0 \end{cases}$$

**Remark 1.6** It is easy to see that the function  $B(t) = t \log(e+t)$  satisfies the hypothesis of the theorem above and thus

$$\mu \left( \{x \in \mathbb{R}^n : \mathcal{M}_{\alpha,B}(f)(x) > t\} \right) \leq C \psi \left[ \int_{\mathbb{R}^n} B \left( \frac{|f(y)|}{t} \right) d\mu(y) \right],$$

for all  $f \in L^B_\mu(\mathbb{R}^d)$ , where  $\psi = [t \log(e + t^{\alpha/n})]^{n/(n-\alpha)}$ . In the context of spaces of homogeneous type this last result was proved in [8].

The following result is a pointwise estimate between the radial fractional type maximal operator associated to a Young function  $B$  and the radial maximal operator associated to a Young  $\psi$  related with  $B$  on non homogeneous spaces.

**Theorem 1.7** *Let  $0 \leq \alpha < n$  and  $1 < p < n/\alpha$ . Let  $\mu$  be an upper Ahlfors  $n$ -dimensional measure. Let  $q$  and  $s$  be defined by  $1/q = 1/p - \alpha/n$  and  $s = 1 + q/p'$ , respectively. Let  $B$  and  $\phi$  be Young functions such that  $\phi^{-1}(t)t^{\alpha/n} \geq C B^{-1}(t)$  and  $\psi(t) = \phi(t^{1-\alpha/n})$ . Then for every measurable function  $f$ , the following inequality*

$$\mathcal{M}_{\alpha,B}(f)(x) \leq C \mathcal{M}_{\psi}(|f|^{p/s})(x)^{1-\alpha/n} \left( \int_{\mathbb{R}^d} |f(y)|^p d\mu(y) \right)^{\alpha/n}$$

*holds in a.e.  $x$ .*

When  $\mu$  is the Lebesgue measure, the result above was proved in [1] (see also [22] for similar multilinear versions and [9] for the case  $B(t) = t$ , both in the euclidean context).

The next theorem gives sufficient conditions on the function  $B$  in order to obtain the boundedness of  $\mathcal{M}_B$  on  $L^p(\mu)$ . In the euclidean context, this result was proved in [17] and in [23] in the framework of spaces of homogeneous type.

**Theorem 1.8** *Let  $\mu$  be an upper Ahlfors  $n$ -dimensional measure. Let  $B$  be a Young function such that  $B \in B_p$ , then,*

$$\mathcal{M}_B : L^p(\mu) \rightarrow L^p(\mu).$$

The following result is a fractional version of Theorem 1.8 and gives a sufficient condition on the function  $B$  that guarantees the continuity of the radial fractional type maximal operator  $\mathcal{M}_{\alpha,B}$  between Lebesgue spaces with non necessary doubling measures.

**Theorem 1.9** *Let  $\mu$  be an upper Ahlfors  $n$ -dimensional measure. Let  $0 < \alpha < n$  and  $1 < p \leq n/\alpha$ . Let  $q$  and  $s$  be defined by  $1/q = 1/p - \alpha/n$  and  $s = 1 + q/p'$  respectively. Let  $B$  be a submultiplicative Young function such that  $B^{q/p} \in B_q$  and let  $\phi$  be a Young function such that  $\phi^{-1}(t)t^{\alpha/n} \geq C B^{-1}(t)$ . Then*

$$\mathcal{M}_{\alpha,B} : L^p(\mu) \rightarrow L^q(\mu).$$

The next theorem is very interesting since it allows us to readily find examples of  $A_1$  weights on non homogeneous spaces.

**Theorem 1.10** *Given  $\alpha$ ,  $0 < \alpha < n$ , and a non-negative function  $f$ . There exists a constant  $C$  such that,*

$$\mathcal{M}(\mathcal{M}_{\alpha}f)(x) \leq C \mathcal{M}_{\alpha}f(x).$$

*Proof:* Fix a cube  $Q$ . We shall see that,

$$\frac{1}{l(Q)^n} \int_Q \mathcal{M}_\alpha f(y) d\mu(y) \leq C \mathcal{M}_\alpha f(x) \quad \text{for a.e. } x \in Q.$$

with  $C$  independent of  $Q$ . Let  $\tilde{Q} = Q^3$ , the 3-dilated of  $Q$ . We write  $f = f_1 + f_2$  with  $f_1 = f\chi_{\tilde{Q}}$ . Then,  $\mathcal{M}_\alpha f(x) \leq \mathcal{M}_\alpha f_1(x) + \mathcal{M}_\alpha f_2(x)$ .

$$\begin{aligned} \frac{1}{l(Q)^n} \int_Q \mathcal{M}_\alpha f_1(y) d\mu(y) &= \frac{1}{l(Q)^n} \int_0^\infty \mu\{x \in Q : \mathcal{M}_\alpha f_1(x) > t\} dt \leq \\ &\leq \frac{1}{l(Q)^n} \left( \mu(Q)R + \int_R^\infty \mu\{x \in Q : \mathcal{M}_\alpha f_1(x) > t\} dt \right) \end{aligned}$$

By [7, Prop 2.1], we know that  $\|\mathcal{M}_\alpha f\|_{L^{\frac{n}{n-\alpha}}(\mu)} \leq \|f\|_{L^1(\mu)}$ . Then, since  $\mu(Q) \leq l(Q)^n$

$$\frac{1}{l(Q)^n} \int_Q \mathcal{M}_\alpha f_1(y) d\mu(y) \leq R + \frac{c}{l(Q)^n} \|f_1\|_{L^1(\mu)}^{n/n-\alpha} \int_R^\infty \frac{dt}{t^{n/n-\alpha}}.$$

By taking  $R = \frac{\|f_1\|_{L^1(\mu)}}{l(Q)^{n-\alpha}}$ , we get

$$\frac{1}{l(Q)^n} \int_Q \mathcal{M}_\alpha f_1(y) d\mu(y) \leq C_{\alpha,n} \frac{\|f_1\|_{L^1(\mu)}}{l(Q)^{n-\alpha}} = \frac{C_{\alpha,n}}{l(\tilde{Q})^{n-\alpha}} \int_{\tilde{Q}} f(y) d\mu(y) \leq C_{\alpha,n} \mathcal{M}_\alpha f(x)$$

for every  $x \in Q$ .

To deal with  $\mathcal{M}_\alpha f_2$  is even simpler. It is enough to see that, because of the fact that  $f_2$  lives far from  $Q$  (outside  $\tilde{Q}$ ), for any two points  $x, y \in Q$ , we have  $\mathcal{M}_\alpha f_2(x) \leq C \mathcal{M}_\alpha f_2(y)$ , with  $C$  an absolute constant. Indeed if  $Q_0$  is a cube containing  $x$  and meeting  $\mathbb{R}^n \setminus \tilde{Q}$ , then  $Q \subset Q_0^3$ , so that

$$\frac{1}{l(Q_0)^{n-\alpha}} \int_{Q_0} f_2(t) d\mu(t) \leq \frac{3^{n-\alpha}}{l(Q_0^3)^{n-\alpha}} \int_{Q_0^3} f_2(t) d\mu(t) \leq 3^{n-\alpha} \mathcal{M}_\alpha f_2(y).$$

Thus

$$\frac{1}{l(Q)^n} \int_Q \mathcal{M}_\alpha f_2(y) d\mu(y) \leq C \frac{\mu(Q)}{l(Q)^n} \mathcal{M}_\alpha f(x) \leq C \mathcal{M}_\alpha f(x)$$

for every  $x \in Q$ . □

## 2 Preliminaries and auxiliar lemmas

A function  $B : [0, \infty) \rightarrow [0, \infty)$  is a Young function if it is convex and increasing, if  $B(0) = 0$ , and if  $B(t) \rightarrow \infty$  as  $t \rightarrow \infty$ . We also deal with submultiplicative Young functions, which means that  $B(st) \leq B(s)B(t)$  for every  $s, t > 0$ . If  $B$  is a submultiplicative Young function, it follows that  $B'(t) \simeq B(t)/t$  for every  $t > 0$

Given a Young function  $B$  and a cube  $Q$ , we define the radial Luxemburg average of  $f$  on  $Q$  associated to  $\mu$  by

$$\|f\|_{B,Q} = \inf \left\{ \lambda > 0 : \frac{1}{l(Q)^n} \int_Q B \left( \frac{|f(x)|}{\lambda} \right) d\mu(x) \leq 1 \right\}. \quad (2.1)$$

The radial Luxemburg average has two rescaling properties which we will use repeatedly. Given any Young function  $A$  and  $r > 0$ ,

$$\|f^r\|_{A,Q} = \|f\|_{B,Q}^r,$$

where  $B(t) = A(t^r)$ . By convexity, if  $\tau > 1$

$$\|f\|_{A,Q} \leq \tau^n \|f\|_{A,\tau Q}.$$

Given a Young function  $B$ , the complementary Young function  $\tilde{B}$  is defined by

$$\tilde{B}(t) = \sup_{s>0} \{st - B(s)\}, \quad t > 0.$$

It is well known that  $B$  and  $\tilde{B}$  satisfy the following inequality

$$t \leq B^{-1}(t) \tilde{B}^{-1} \leq 2t.$$

It is also easy to check that the following version on the Hölder's inequality

$$\frac{1}{l(Q)^n} \int_Q |f(x)g(x)| d\mu(x) \leq 2 \|f\|_{B,Q} \|g\|_{\tilde{B},Q}$$

holds. Moreover, there is a further generalization of the inequality above. If  $A$ ,  $B$  and  $C$  are Young functions such that for every  $t \geq t_0 > 0$ ,

$$B^{-1}(t)C^{-1}(t) \leq c A^{-1}(t),$$

then, the inequality

$$\|fg\|_{A,Q} \leq K \|f\|_{B,Q} \|g\|_{C,Q} \quad (2.2)$$

holds.

In this papers we give boundedness results for the maximal operator  $\mathcal{M}_{\alpha,B}$  between Lebesgue spaces. We begin with an usefull property related with  $B_p$  condition.

**Proposition 2.1** *Let  $B$  be a submultiplicative Young function and let  $\phi$  be a Young function such that  $\phi^{-1}(t)t^{\alpha/n} \geq C B^{-1}(t)$ . Let  $1 < p < n/\alpha$ ,  $1/q = 1/p - \alpha/n$  and  $s = q(1 - \alpha/n)$ . If  $B^{q/p} \in B_q$ , then the function  $\psi$  defined by  $\psi(t) = \phi(t^{1-\alpha/n})$  belongs to  $B_s$ .*

*Proof:* From the definition of  $\psi$  and by changing variables we obtain that

$$\int_1^\infty \frac{\psi(t)}{t^s} \frac{dt}{t} = \int_1^\infty \frac{\phi(t^{1-\alpha/n})}{t^s} \frac{dt}{t} = \left( \frac{n}{n-\alpha} \right) \int_1^\infty \frac{\phi(r)}{r^{ns/(n-\alpha)}} \frac{dr}{r}.$$



From the relation between  $B$  and  $\phi$  it is easy to see that  $\phi$  is a submultiplicative function. Thus, noting that  $q = ns/(n - \alpha)$  we obtain

$$\begin{aligned} \int_1^\infty \frac{\phi(r)}{r^{ns/(n-\alpha)}} \frac{dr}{r} &= \int_1^\infty \frac{\phi(r)}{r^q} \frac{dr}{r} \\ &\leq c \int_c^\infty \frac{u^{1+q\alpha/n}}{(\phi^{-1}(u)u^{\alpha/n})^q} \frac{du}{u} \\ &\leq C \int_c^\infty \frac{u^{q/p}}{B^{-1}(u)^q} \frac{du}{u} = C \int_c^\infty \frac{B(t)^{q/p}}{t^q} \frac{dt}{t} < \infty. \end{aligned}$$

□

The proof of Theorem 1.5 requires any lemmas. The first of them was proved in [5] and the second in [10].

**Lemma 2.2** *Given  $0 \leq \alpha < n$ , let  $B$  be a Young function such that for  $\alpha > 0$ ,  $B(t)/t^{n/\alpha}$  is decreasing for all  $t > 0$ . Then the function  $\phi_1$  in Theorem 1.5 is increasing, and  $\phi_1(s)/s$  is decreasing. Moreover, there exists  $\phi$  such that  $\phi(s) \leq C_1\phi_1(s)$  and  $\phi$  is invertible.*

**Lemma 2.3** *If  $\phi(t)/t$  is decreasing, then for any positive sequence  $\{x_j\}$ ,*

$$\phi\left(\sum_j x_j\right) \leq \sum_j \phi(x_j).$$

The following lemma is a generalization of Lemma 3.2 in [7] for radial Luxemburg type averages. When  $\mu$  is the Lebesgue measure, it was proved in [5].

**Lemma 2.4** *Suppose that  $0 \leq \alpha < n$ ,  $B$  is a Young function and  $f$  is a nonnegative bounded function with compact support. If for  $t > 0$  and any cube  $Q$*

$$l(Q)^\alpha \|f\|_{B,Q} > t,$$

*then, there exist a dyadic cube  $P$  such that  $Q \subset 3P$  satisfying*

$$l(P)^\alpha \|f\|_{B,P} > \beta t,$$

*where  $\beta$  is a nonnegative constant.*

*Proof:* Let  $Q$  be a cube with

$$l(Q)^\alpha \|f\|_{B,Q} > t. \tag{2.3}$$

Let  $k$  be the unique integer such that  $2^{-(k+1)} < l(Q) \leq 2^{-k}$ . There is some dyadic cube with side length  $2^{-k}$ , and at most  $2^d$  of them,  $\{J_i : i = 1, \dots, N\}$ ,  $N \leq 2^d$ , meeting the interior of  $Q$ . It is easy to see that for one of these cubes, say  $J_1$ ,

$$\frac{t}{2^d} < l(Q)^\alpha \|\chi_{J_1} f\|_{B,Q}.$$

This can be seen as follows. If for each  $i = 1, 2, \dots, N$  we have

$$l(Q)^\alpha \|\chi_{J_i} f\|_{B,Q} \leq \frac{t}{2^d},$$

since  $Q \subset \cup_{i=1}^N J_i$  we obtain that

$$\begin{aligned} l(Q)^\alpha \|f\|_{B,Q} &= l(Q)^\alpha \|\chi_{\cup_{i=1}^N J_i} f\|_{B,Q} \\ &\leq l(Q)^\alpha \sum_{i=1}^N \|\chi_{J_i} f\|_{B,Q} \leq N \frac{t}{2^d} \leq t, \end{aligned}$$

contradicting (2.3). Using that  $l(Q) \leq l(J_1) < 2l(Q)$  we can also show that

$$\frac{t}{2^d} < l(Q)^\alpha \|\chi_{J_1} f\|_{B,Q} \leq 2^n l(J_1)^\alpha \|f\|_{B,J_1}$$

and  $Q \subset 3J_1$ . □

### 3 Proof of the main results

#### Proof of Theorem (1.5).

Fix a non-negative function  $f \in L_\mu^B(\mathbb{R}^d)$ . Fix  $t > 0$  and define

$$E_t = \{x \in \mathbb{R}^d : \mathcal{M}_{\alpha,B} f(x) > t\}.$$

If  $t$  is such that the set  $E_t$  is empty, we have nothing to prove. Otherwise, for each  $x \in E_t$  there exists a cube  $Q_x$  containing  $x$  such that

$$l(Q_x)^\alpha \|f\|_{B,Q_x} > t.$$

By Lemma 2.4, there exists a constant  $\beta$  and a dyadic cube  $P_x$  with  $Q_x \subset 3P_x$  such that

$$l(P_x)^\alpha \|f\|_{B,P_x} > \beta t. \tag{3.1}$$

Since  $f \in L_\mu^B(\mathbb{R}^d)$ , it is not hard to prove that we can replace the collection  $\{P_x\}$  with a maximal disjoint subcollection  $\{P_j\}$ . Each  $P_j$  satisfies (3.1) and, by our choice of the  $Q_x$ 's,  $E_t \subset \cup_j 3P_j$ . By Lemmas 2.2 and 2.3,

$$\phi_1(\mu(E_t)) \leq \sum_j \phi_1(\mu(3P_j)).$$

On the other hand, inequality (3.1) implies that, for each  $j$ ,

$$\frac{1}{l(P_j)^n} \int_{P_j} B \left( \frac{l(P_j)^\alpha |f|}{\beta t} \right) d\mu > 1,$$

and then by the definition and properties of  $h_B$ ,

$$\begin{aligned}
1 &< \frac{1}{l(P_j)^n} \int_{P_j} B \left( \frac{3^\alpha l(P_j)^\alpha |f(x)|}{3^\alpha \beta t} \right) d\mu(x) \\
&\leq \frac{3^n h_B(3^{-\alpha} \beta^{-1}) h_B(l(3P_j)^\alpha)}{l(3P_j)^n} \int_{P_j} B \left( \frac{|f(x)|}{t} \right) d\mu(x) \\
&\leq \frac{C}{\phi_1(l(3P_j)^n)} \int_{P_j} B \left( \frac{|f(x)|}{t} \right) d\mu(x).
\end{aligned}$$

Hence, since the  $P_j$ 's are disjoint,

$$\begin{aligned}
\sum_j \phi_1(\mu(3P_j)) &\leq \sum_j \phi_1(l(3P_j)^n) \\
&\leq C \sum_j \int_{P_j} B \left( \frac{|f(x)|}{t} \right) d\mu(x) \\
&\leq C \int_{\mathbb{R}^d} B \left( \frac{|f(x)|}{t} \right) d\mu(x).
\end{aligned}$$

□

### Proof of Theorem 1.2.

Without loss of generality, we can assume that  $f$  is a non-negative bounded function with compact support. This guarantees that  $\mathcal{M}_{\alpha,B}f$  is finite  $\mu$ -almost everywhere. In fact,  $f \in L_\mu^{p_0}(\mathbb{R}^d)$  where  $p_0$  is the exponent of the hypotheses. From Theorem 1.9 we get that  $\mathcal{M}_{\alpha,B}f \in L_\mu^{q_0}(\mathbb{R}^d)$  and thus

$$\mathcal{M}_{\alpha,B}f(x) < \infty \quad \text{a.e. } x \in \mathbb{R}^d.$$

For each  $k \in \mathbb{Z}$  let  $\Omega_k = \{x \in \mathbb{R}^d : 2^k < \mathcal{M}_{\alpha,B}f(x) \leq 2^{k+1}\}$ . Thus

$$\mathbb{R}^d = \cup_{k \in \mathbb{Z}} \Omega_k.$$

Then, for every  $k$  and every  $x \in \Omega_k$ , by the definition of  $\mathcal{M}_{\alpha,B}f$ , there is a cube  $Q_x^k$  containing  $x$ , such that

$$l(Q_x^k)^\alpha \|f\|_{B, Q_x^k} > 2^k,$$

and, from Lemma 2.4 there exist a constant  $\beta$  and a dyadic cube  $P_x^k$  with  $Q_x^k \subset 3P_x^k$  such that

$$l(P_x^k)^\alpha \|f\|_{B, P_x^k} > \beta 2^k. \quad (3.2)$$

From the fact that  $B$  is submultiplicative and  $\text{supp}(f)$  is a compact set, the inequality above allow us to obtain that

$$\frac{l(P_x^k)^n}{B(l(P_x^k)^\alpha)} < \int_{P_x^k} B \left( \frac{|f|}{2^k \beta} \right) d\mu \leq C \mu(\text{supp}(f)) \leq C.$$

From the hypotheses on  $B$  it is easy to check that

$$C_1 \varphi^{-1} \left( \frac{l(P_x^k)^n}{C} \right) \left( \frac{l(P_x^k)^n}{C} \right)^{\alpha/n} \leq B^{-1} \left( \frac{l(P_x^k)^n}{C} \right) \leq l(P_x^k)^\alpha,$$

which allow us to conclude that, for each  $k$ ,  $l(P_x^k)$  is bounded by a constant independent of  $x$ . Then, there is a subcollection of maximal cubes (and so disjoint)  $\{P_j^k\}_j$  such that every  $Q_x^k$  is contained in  $3P_j^k$  for some  $j$  and, as a consequence,  $\Omega_k \subset_j 3P_j^k$ . Next, decompose  $\Omega_k$  into the sets

$$E_1^k = 3P_1^k \cap \Omega_k, E_2^k = (3P_2^k \setminus 3P_1^k) \cap \Omega_k, \dots, E_j^k = (3P_j^k \setminus \cup_{r=1}^{j-1} 3P_r^k) \cap \Omega_k, \dots$$

Then

$$\mathbb{R}^d = \bigcup_{k \in \mathbb{Z}} \Omega_k = \bigcup_{j,k} E_j^k$$

and these sets are pairwise disjoint. Let  $K$  be a fixed positive integer which will go to infinity later, and let  $\Lambda_K = \{(j, k) \in \mathbb{N} \times \mathbb{Z} : |k| \leq K\}$ . By using that  $E_j^k \subset \Omega_k$  and that the cubes  $P_j^k$  satisfy (3.2) we obtain that

$$\begin{aligned} \mathcal{I}_k &= \int_{\cup_{-K}^K \Omega_k} (\mathcal{M}_{\alpha,B} f(x))^q u(x) d\mu(x) \\ &= \sum_{(j,k) \in \Lambda_k} \int_{E_j^k} (\mathcal{M}_{\alpha,B} f(x))^q u(x) d\mu(x) \\ &\leq \sum_{(j,k) \in \Lambda_k} u(E_j^k) 2^{(k+1)q} \\ &\leq C 2^q \sum_{(j,k) \in \Lambda_k} u(E_j^k) \left( l(P_j^k)^\alpha \|f\|_{B, P_j^k} \right)^q \\ &\leq C 2^q \sum_{(j,k) \in \Lambda_k} u(3P_j^k) \left( l(P_j^k)^\alpha \|f v^{1/p}\|_{C, P_j^k} \|v^{-1/p}\|_{A, P_j^k} \right)^q, \end{aligned}$$

where in the last inequality we have used the generalized Hölder's inequality and the hypothesis on the functions  $A$ ,  $B$  and  $C$ . Now, by applying the hypothesis on the weights we obtain that

$$\mathcal{I}_k \leq C \sum_{(j,k) \in \Lambda_k} l(3P_j^k)^{nq/p} \|f v^{1/p}\|_{C, P_j^k}^q = C \int_{\mathcal{Y}} T_k(f v^{1/p})^q d\nu,$$

where  $\mathcal{Y} = \mathbb{N} \times \mathbb{Z}$ ,  $\nu$  is de measure in  $\mathcal{Y}$  given by  $\nu(j, k) = l(3P_j^k)^{nq/p}$  and, for every measurable function  $h$ , the operator  $T_k$  is defined by the expression

$$T_k h(j, k) = \|\varphi\|_{C, P_j^k} \chi_{\Lambda_k}(j, k).$$

Then, if we prove that  $T_k : L^p(\mathbb{R}^d, \mu) \rightarrow L^q(\mathcal{Y}, \nu)$  is bounded independently of  $K$ , we shall obtain that

$$\mathcal{I}_k \leq C \int_{\mathcal{Y}} T_k(f v^{1/p})^q d\nu \leq C \left( \int_{\mathbb{R}^d} (f v^{1/p})^p d\mu \right)^{q/p} = C \left( \int_{\mathbb{R}^d} f^p v d\mu \right)^{q/p},$$

and we shall get the desired inequality by doing  $K \rightarrow \infty$ . But the proof of the boundedness of  $T_k$  follows the same arguments as in Theorem 5.3 in [7], by using now that the function  $C \in B_p$ , so we omit it.  $\square$

**Proof of Theorem 1.7.** Let  $g(x) = |f(x)|^{p/s}$ , then

$$|f(x)| = g(x)^{s/p+\alpha/n-1} g(x)^{1-\alpha/n}.$$

Let  $x \in \mathbb{R}^d$  and  $Q$  be a fixed cube containing  $x$ . By the generalized Hölder's inequality (2.2) and the fact that

$$g(x)^{(s/p+\alpha/n-1)n/\alpha} = |f|^p,$$

we get

$$\begin{aligned} l(Q)^\alpha \|f\|_{B,Q} &\leq C l(Q)^\alpha \|g^{1-\alpha/n}\|_{\phi,Q} \|g^{s/p+\alpha/n-1}\|_{n/\alpha,Q} \\ &= C l(Q)^\alpha \|g\|_{\psi,Q}^{1-\alpha/n} \left( \frac{1}{l(Q)^n} \int_Q |f(y)|^p d\mu(y) \right)^{\alpha/n} \\ &\leq C [\mathcal{M}_\psi(g)(x)]^{1-\alpha/n} \|f\|_{L^p(\mu)}^{p\alpha/n}. \end{aligned}$$

$\square$

**Proof of Theorem 1.8.** From Theorem 1.5 applied to the case  $\alpha = 0$  it is easy to check that

$$\mu(\{y \in \mathbb{R}^d : \mathcal{M}_B f(y) > 2t\}) \leq C \int_{|f|>t} B(|f|/t) d\mu(t).$$

Thus, by changing variables and using inequality above we obtain that

$$\begin{aligned} \int_{\mathbb{R}^d} \mathcal{M}_B f(y)^p d\mu(y) &= C \int_0^\infty t^p \mu(\{y \in \mathbb{R}^d : \mathcal{M}_B f(y) > 2t\}) \frac{dt}{t} \\ &\leq C \int_{\mathbb{R}^d} \int_0^{|f(y)|} t^p B\left(\frac{|f(y)|}{t}\right) \frac{dt}{t} d\mu(y) \\ &= C \left( \int_{\mathbb{R}^d} |f(y)|^p d\mu(y) \right) \left( \int_1^\infty \frac{B(s)}{s^p} \frac{ds}{s} \right). \end{aligned}$$

Thus, condition  $B_p$  allow us to obtain the desired result.  $\square$

**Proof of Theorem 1.9.** By Theorem 1.7, if  $1 < p < n/\alpha$ , we have

$$\begin{aligned} \left( \int_{\mathbb{R}^d} (\mathcal{M}_{\alpha,B}(f))^q d\mu \right)^{1/q} &\leq C \left( \int_{\mathbb{R}^d} \left( \mathcal{M}_\psi(|f|^{p/s})^{1-\alpha/n} \|f\|_{L^p(\mu)}^{p\alpha/n} \right)^q d\mu \right)^{1/q} \\ &= C \|f\|_{L^p(\mu)}^{p\alpha/n} \left( \int_{\mathbb{R}^d} \mathcal{M}_\psi(|f|^{p/s})^s d\mu \right)^{1/q}. \end{aligned}$$

From Proposition 2.1 we have that the function  $\psi \in B_s$ . Thus Theorem 1.8 implies that  $\mathcal{M}_\psi : L^s(\mu) \rightarrow L^s(\mu)$ , and thus

$$\left( \int_{\mathbb{R}^d} (\mathcal{M}_{\alpha,B}(f))^q d\mu \right)^{1/q} \leq C \|f\|_{L^p(\mu)}^{p\alpha/n} \left( \int_{\mathbb{R}^d} (|f|^{p/s})^s d\mu \right)^{1/q} = C \|f\|_{L^p(\mu)}.$$

On the other hand, if  $p = n/\alpha$  and  $Q$  is a cube such that  $x \in Q$  we obtain that

$$\begin{aligned} l(Q)^\alpha \|f\|_{n,Q} &\leq Cl(Q)^\alpha \|\chi_Q\|_{\phi,Q} \|f\|_{n/\alpha,Q} \\ &\leq C \|f\|_{n/\alpha}, \end{aligned}$$

and thus

$$\mathcal{M}_{\alpha,B}(f)(x) \leq C \|f\|_{n/\alpha}$$

for a.e.  $x$  which leads us with the desired result.  $\square$

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*Gladis Pradolini, Instituto de Matemática Aplicada del Litoral (CONICET-UNL), Departamento de Matemática (FIQ-UNL), 3000 Santa Fe, Argentina. e-mail address: gpradolini@santafe-conicet.gov.ar*

*Jorgelina Recchi, Instituto de Matemática Bahía Blanca (CONICET-UNS) Departamento De Matemáticas, Universidad Nacional Del Sur, 8000 Bahía Blanca, Argentina. e-mail address: drecchi@uns.edu.ar*